Studies of two β-decay correlation coefficients

I. S. Towner and J. C. Hardy

In studies of nuclear beta decay, the two most commonly examined correlation coefficients are the beta-neutrino angular correlation coefficient, $a_{e\nu}$, and the beta asymmetry, A_β the correlation between the direction of polarization in the parent nucleus and the direction of the emitted electron. These two correlation coefficients can be written in the form

$$a_{e\nu}(W) = a_{e\nu}^0 + \Delta a_{e\nu}(W),$$
$$A_\beta(W) = A_\beta^0 + \Delta A_\beta(W), \tag{1}$$

where $a_{e\nu}^0$ and A_β^0 are the major contributions depending on just two parameters, a_1 and c_1, defined as $a_1 = g_V \mathcal{M}_F$ and $c_1 = g_A \mathcal{M}_{GT}$ with $\mathcal{M}_F$ and $\mathcal{M}_{GT}$ being the Fermi and Gamow-Teller matrix elements and g_V and g_A their respective coupling constants:

$$a_{e\nu}^0 = \frac{a_1^2 - \frac{1}{3}c_1^2}{a_1^2 + c_1^2} \qquad A_\beta^0 = \frac{2\alpha a_1 c_1 \pm \gamma c_1^2}{a_1^2 + c_1^2}. \tag{2}$$

Here α and γ are simple geometric functions of the initial and final nuclear spins, J_i and J_f, and defined as

$$\alpha = (-)^{J_i - J_f} W(J_i J_i 01; 1 J_f) \left[\frac{3J_i(2J_i+1)}{(J_i+1)}\right]^{1/2} = \delta_{J_i J_f}\left[\frac{J_i}{J_i+1}\right]^{1/2}, \tag{3}$$

$$\gamma = (-)^{J_i - J_f + 1} W(J_i J_i 11; 1 J_f) \left[\frac{6J_i(2J_i+1)}{(J_i+1)}\right]^{1/2}, \tag{4}$$

with $W(...)$ being a Racah recoupling coefficient. The upper sign is used for electron emission, the lower sign for positron emission. Added to these major contributions in Eq. (1) are small correction terms $\Delta a_{e\nu}$ and ΔA_β, typically of order 1%, that are dependent on the electron energy, W. Our goal here is to study these correction terms firstly by using the exact *β*-decay formalism of Behrens-Bühring (BB) [1], secondly by finding approximate formulae based on BB obtained by making appropriate expansions of the lepton functions, and thirdly by using approximate formulae given in Holstein [2] and Wilkinson [3]. The formalism associated with this study is given in Towner's summer report [4] and will not be reproduced here.

In what follows we give some numerical results for the beta-neutrino correlation coefficient $a_{e\nu}$ and the *β*-asymmetry parameter A_β for the neutron and five examples of mirror transitions in the *s,d*-shell. These five examples were the cases studied by Naviliat-Cuncic and Severijns [5], who used measured values of lifetimes and correlation coefficients to determine the Cabibbo-Kobayashi-Maskawa quark-mixing matrix element, V_{ud}. In their analysis, these authors used the Holstein formulae to apply corrections $\Delta a_{e\nu}(W)$ and $\Delta A_{\beta}(W)$ to the measured correlation coefficients, Eq. (1). In this work, we will display the numerical differences between the BB and Holstein formulae for these corrections.

The beta-neutrino and beta-asymmetry correlation coefficients can be cast into the form

$$a(W) = a^0(1 + s_0 + s_1 W + \frac{s_2}{W} + s_3 W^2), \qquad (5)$$

where a stands for either the beta-neutrino correlation coefficient $a_{e\nu}$ or the beta-asymmetry parameter A_β and a^0 its Standard-Model value, either $a_{e\nu}^0$ or A_β^0 as given in Eq.(2).

Eight nuclear-structure parameters a_1, c_1, x, $\bar{b}$, $\bar{d}$, $\bar{g}$, $\bar{J_2}$ and $\bar{p}$ need to be specified. They are defined as

$$\begin{aligned}
a_1 &= g_V \mathcal{M}_F, \\
c_1 &= g_A \mathcal{M}_{GT}, \\
x &= -\sqrt{10}\frac{\mathcal{M}_{1y}}{\mathcal{M}_{\sigma r^2}}, \\
\bar{b} &= \frac{1}{MR}\left[\frac{g_M}{g_A} + \frac{g_V}{g_A}\frac{\mathcal{M}_L}{\mathcal{M}_{GT}}\right], \\
\bar{d} &= \frac{1}{MR}\frac{\mathcal{M}_{\sigma L}}{\mathcal{M}_{GT}}, \\
\bar{g} &= -\frac{4}{3}\sqrt{6}\frac{1}{R^2} g_V \mathcal{M}_Q, \\
\bar{J_2} &= -2\frac{1}{R^2} g_A \mathcal{M}_{2y}, \\
\bar{p} &= \frac{1}{(MR)^2}\frac{g_P}{g_A},
\end{aligned} \qquad (6)$$

with $g_V = 1$, the vector coupling constant, g_A the axial-vector coupling constant, $g_M = 4.706$ the nucleon isovector magnetic moment, M the nucleon mass in electron rest-mass units and R the nuclear radius in electron Compton wavelength units. The pseudoscalar coupling constant, g_P, is fixed from the PCAC relation at zero momentum transfer: $g_P = -(2M/m_\pi)^2 g_A$, with m_π the pion mass. The required nuclear matrix elements are defined in Eq. (68) of [2]. Schematically they are written: $\mathcal{M}_F = \langle 1 \rangle$, $\mathcal{M}_{GT} = \langle \sigma \rangle$, $\mathcal{M}_{\sigma r^2} = \langle r^2 \sigma \rangle$, $\mathcal{M}_{ky} = (16\pi/5)^{1/2}\langle r^2 [Y_2 \times \sigma]^{(k)} \rangle$, $\mathcal{M}_L = \langle L \rangle$, $\mathcal{M}_{\sigma L} = \langle \sigma \times L \rangle$ and $\mathcal{M}_Q = (4\pi/5)^{1/2}\langle r^2 Y_2 \rangle$. For the present time, we have left out the relativistic matrix elements denoted in [2] as $\mathcal{M}_{r.p}$, $\mathcal{M}_{\{r,p\}}$ and $\mathcal{M}_{\sigma rp}$ and dropped any second-class current terms.

For the neutron and the five mirror transitions in the s,d-shell, a_1 is fixed by the CVC hypothesis to be the same in all cases, $a_1 = 1$. The matrix elements, $\mathcal{M}_{\sigma L}$ and $\mathcal{M}_{2y}$ and hence $\bar{d}$ and $\bar{J_2}$, are zero because the expectation values of $\boldsymbol{\sigma} \times \mathbf{L}$ and $[\boldsymbol{\sigma} \times Y_2]^{(2)}$ vanish in diagonal matrix elements. The neutron is considered a pure S-state, so $\mathcal{M}_L$, $\mathcal{M}_Q$, $\mathcal{M}_{1y}$, and hence x, are all zero, while the Gamow-Teller matrix element is fixed at $\mathcal{M}_{GT} = \sqrt{3}$. For the s,d-shell nuclei, we perform a shell-model calculation using the USD effective interaction [6] to determine the nuclear matrix elements. Their values are given in Table I. Coupling constants g_A and g_M in finite nuclei are customarily treated as effective coupling constants because they are combined with nuclear matrix elements that have been calculated, inevitably, in a finite model space. Adjusting the coupling constants is one way of compensating for the inadequacy of the shell-model calculation. In practice, we adjust g_A and g_M so that the shell-model calculation yields the experimental value of the partial-decay lifetime and the isovector combination of magnetic moments. Values of g_A^{eff} and g_M^{eff} are given in Table I. We also need to specify the nuclear

Table I. Nuclear matrix elements and related parameters used in the computations of a_{ev} and A_β. For the *s,d*-nuclei, the matrix elements were obtained with the shell model using the USD effective interaction [6]. Coupling constants g_A and g_M are quenched in finite nuclei in order that the shell model reproduces the experimental values for the partial decay lifetimes and the isovector combination of magnetic moments. For these mirror transitions, $\bar{\boldsymbol{d}}$ and $\bar{\boldsymbol{J_2}}$ are identically zero.

Decay Nucleus	$\mathcal{M}_{GT}$	$\mathcal{M}_{\sigma r^2}$ fm^2	$\mathcal{M}_L$	$\mathcal{M}_{1y}$ fm^2	g_A^{eff}	g_M^{eff}	a_1	c_1	x	$\bar{b}$	$\bar{g}$	$\bar{p}$
n	1.732	5.39	0.000	0.00	1.270	4.706	1.00	2.200	0.000	0.700	0.000	−6.456
^{19}Ne	−1.676	−18.22	−0.717	−0.15	0.951	4.237	1.00	−1.593	−0.025	0.275	0.000	−0.569
^{21}Na	0.726	8.09	0.944	0.92	0.969	4.919	1.00	0.703	−0.360	0.351	−0.263	−0.540
^{29}P	0.513	6.21	0.556	4.89	1.015	4.957	1.00	0.521	−2.491	0.315	0.000	−0.506
^{35}Ar	0.328	4.17	−1.493	8.51	0.867	3.806	1.00	0.284	−6.461	−0.042	0.048	−0.418
^{37}K	−0.624	−8.05	1.416	−10.11	0.942	4.220	1.00	−0.587	−3.972	0.100	0.144	−0.425

radius parameter, R. We set $R^2 = \frac{3}{5}\langle r^2\rangle$, where $\langle r^2\rangle$is the mean-square radius of the charge-density distribution of the daughter nucleus in β-decay. This is the charge distribution the emitted β-decay electron encounters and is used in the Dirac equation that is solved to determine the electron wave function.

Our numerical results for the electron-neutrino correlation and beta-asymmetry coefficients are given in Tables II and III. The corrections $\Delta a_{ev}(W)$ and $\Delta A_\beta(W)$ have been averaged over the entire electron spectrum $\overline{a_{ev}}$ and $\overline{A_\beta}$. For the neutron, this average correction is about 3% for a_{ev} and 2% for A_β. The formulae of Holstein [2] slightly underestimate this correction compared to the exact result computed from the formulae of Behrens and Bühring [1]. The results obtained by Wilkinson [3] are derivable from Holstein's formulae on setting $(\alpha Z) \to 0$ and $R^2 \to 0$, but retaining terms linear in R. Only the weak magnetism term in $\bar{b}$ and the kinematic recoil correction are retained by Wilkinson. We see that Wilkinson's results for the neutron are very similar to Holstein's.

For the five examples of mirror transitions in the *s,d*-shell, we see that Holstein's formulae significantly underestimate the correction to a_{ev} in ^{19}Ne, ^{21}Na, ^{29}P and to A_β in ^{21}Na and ^{29}P, while overestimating the correction to A_β in ^{19}Ne. This discrepancy can be traced to the lack of a term in $(\alpha Z)\bar{b}$, an electromagnetic correction in the weak-magnetism form factor. For ^{35}Ar and ^{37}K, the difference between Holstein and BB is somewhat less, but then from Table I one observes the weak-magnetism parameter $\bar{b}$ is somewhat less in these two cases.

Table II. Electron-neutrino correlation coefficient $\boldsymbol{a_{ev}^0}$ from the Standard Model, the correction to it, $\boldsymbol{\Delta a_{ev}(W)}$, averaged over the electron energy spectrum $\overline{\boldsymbol{\Delta a_{ev}}}$ and the final corrected coefficient again averaged over the entire energy spectrum $\overline{\boldsymbol{a_{ev}}}$. Shown are the results obtained with shell-model nuclear matrix elements given in Table I from the approximate formulae of Wilkinson [3], Holstein [2], Behrens-Bühring (BB) [1], and the exactly computed result with the BB formalism. Also given are the parameters of the correction s_0, s_1, s_2 and s_3 as defined in Eq. (5).

Nucleus		$s_0(\%)$	$s_1(\%)$	$s_2(\%)$	$s_3(\%)$	$a_{e\nu}^0$	$\overline{\Delta a_{e\nu}}$	$\overline{a_{e\nu}}$
neutron	Wilkinson	−3.052	3.449	0.141	0.000	−0.1050	−0.0026	−0.1076
	Holstein	−3.043	3.453	0.141	−0.003	−0.1050	−0.0027	−0.1077
	BB-approx	−2.311	3.456	0.142	−0.013	−0.1050	−0.0034	−0.1084
	BB-exact	−2.509	3.510	0.139	−0.013	−0.1050	−0.0033	−0.1083
^{19}Ne	Holstein	−14.85	5.219	−0.163	0.087	0.0435	0.0004	0.0439
	BB-approx	−23.65	5.231	−0.136	0.107	0.0435	−0.0033	0.0402
	BB-exact	−23.11	5.129	−0.096	0.104	0.0435	−0.0032	0.0403
^{21}Na	Holstein	−1.175	0.364	−0.120	0.007	0.5587	0.0000	0.5587
	BB-approx	−2.242	0.363	−0.086	0.007	0.5587	−0.0058	0.5529
	BB-exact	−2.226	0.365	−0.083	0.007	0.5587	−0.0057	0.5530
^{29}P	Holstein	−0.830	0.154	−0.108	0.006	0.7154	−0.0003	0.7151
	BB-approx	−1.595	0.148	−0.058	0.006	0.7154	−0.0058	0.7096
	BB-exact	−1.597	0.154	−0.056	0.006	0.7154	−0.0056	0.7098
^{35}Ar	Holstein	0.131	−0.070	−0.084	0.005	0.9004	−0.0010	0.8995
	BB-approx	0.218	−0.077	−0.015	0.005	0.9004	−0.0004	0.9001
	BB-exact	0.258	−0.073	−0.015	0.005	0.9004	0.0002	0.9006
^{37}K	Holstein	−0.224	−0.021	−0.103	0.008	0.6580	−0.0004	0.6575
	BB-approx	−0.517	−0.038	−0.033	0.009	0.6580	−0.0028	0.6552
	BB-exact	−0.349	−0.035	−0.033	0.009	0.6580	−0.0016	0.6564

Table III. Beta-asymmetry correlation coefficient A_β^0 from the Standard Model, the correction to it $\Delta A_\beta(W)$ averaged over the electron energy spectrum $\overline{\Delta A_\beta}$ and the final corrected coefficient, again averaged over the entire energy spectrum $\overline{A_\beta}$. Shown are the results obtained with shell-model nuclear matrix elements given in Table I from the approximate formulae of Wilkinson [3], Holstein [2], Behrens-Bühring (BB) [1], and the exactly computed result with the BB formalism. Also given are the parameters of the correction s_0, s_1, s_2 and s_3 as defined in Eq. (5).

Nucleus		$s_0(\%)$	$s_1(\%)$	$s_2(\%)$	$s_3(\%)$	A_β^0	$\overline{\Delta A_\beta}$	$\overline{A_\beta}$
neutron	Wilkinson	−0.874	1.408	0.141	0.000	−0.1175	−0.0017	−0.1192
	Holstein	−0.871	1.409	0.141	−0.001	−0.1175	−0.0017	−0.1192
	BB-approx	−0.050	1.411	0.142	0.001	−0.1175	−0.0027	−0.1202
	BB-exact	−0.048	1.375	0.142	0.001	−0.1175	−0.0026	−0.1201
^{19}Ne	Holstein	−5.14	3.961	−0.163	0.030	−0.0417	−0.0027	−0.0444
	BB-approx	−13.75	3.972	−0.136	0.028	−0.0417	0.0009	−0.0408
	BB-exact	−13.57	4.079	−0.112	0.027	−0.0417	0.0007	−0.0410
^{21}Na	Holstein	0.308	−0.019	−0.120	−0.001	0.8614	0.0017	0.8631
	BB-approx	0.817	−0.029	−0.086	0.000	0.8614	0.0060	0.8674
	BB-exact	0.735	−0.019	−0.086	0.000	0.8614	0.0056	0.8670
^{29}P	Holstein	0.666	−0.098	−0.108	−0.001	0.6154	0.0011	0.6166
	BB-approx	1.650	−0.094	−0.058	0.000	0.6154	0.0075	0.6229
	BB-exact	1.512	−0.091	−0.059	0.000	0.6154	0.0068	0.6222
^{35}Ar	Holstein	−0.309	0.053	−0.084	−0.001	0.4371	−0.0003	0.4368
	BB-approx	−0.762	0.061	−0.015	0.000	0.4371	−0.0019	0.4352
	BB-exact	−1.239	0.073	−0.014	0.000	0.4371	−0.0037	0.4334
^{37}K	Holstein	0.090	0.013	−0.103	0.002	−0.5739	−0.0012	−0.5752
	BB-approx	0.175	0.016	−0.033	0.003	−0.5739	−0.0021	−0.5760
	BB-exact	0.024	0.025	−0.033	0.003	−0.5739	−0.0015	−0.5754

[1] H. Behrens and W. Bühring, *Electron Radial Wave Functions and Nuclear Beta-decay* (Clarendon Press, Oxford,1982); H. Behrens, H. Genz, M. Conze, H. Feldmeir, W. Stock and A. Richter, Ann. of Phys. **115**, 276 (1978).

[2] B.R. Holstein, Rev. Mod. Phys. 46, 789 (1974); erratum, Rev. Mod. Phys. 48}, 673 (1976).

[3] D.H. Wilkinson, Nucl. Phys. **A377**, 474 (1982).

[4] I.S. Towner, *Parameterizations useful in nuclear beta decay*, private communication.

[5] O. Naviliat-Cuncic and N. Severijns, Phys. Rev. Lett. **102**, 142302 (2009).

[6] B.H. Wildenthal, in *Progress in Particle and Nuclear Physics*, ed. D.H. Wilkinson (Pergamon Press, Oxford 1984) Vol. **11**, p. 5.